PMM U.S.S.R., Vol.53, No.2, pp.250-260, 1989
Printed in Great Britain

0021-8928/89 $10.00+0.00
©1990 Pergamon Press plc

ON THE FORM OF THE PLASTICITY FUNCTIONAL IN ENDOCHRONIC INELASTICITY THEORIES*

O.YU. DINARIYEV and A.B. MOSOLOV

Different modifications of the selection of the plasticity function are considered for plasticity theories of endochronic type, without using the yield surface concept.

Classical plasticity theories (the flow theory type) based on the yield surface concept do not satifactorily describe complex loading experiments. Consequently, a number of authors attempted to omit using the yield surface concept and the unloading conditions when constructing the governing relationships of plasticity theory. Apparently, the best known representatives of this class of theories are the endochronic theory of inelasticity (Valanis) and the tensor-parametric theory of plasticity (Kadashevich et al.). However, interest in plasticity theories that do not utilize the yield surface concept arose much earlier. The formulation of a theory in which there was actually no need to use the yield surface when constructing the governing relationships /1-3/ was even contained in researches of A.A. Il'yushin proposing general principles for constructing a phenomenological theory of plasticity under complex loading conditions.

Let the relation between the stress vector σ and the strain vector e be given for initially isotropic materials in the form of a single-valued O (5)-invariant functional satisfying the retardation property (damped memory)

$$\sigma = \mathbf{F}\{s, \mathbf{e}(\cdot)]_0^s, \quad ds = |de| \tag{1}$$

A specific form of the functional (1) was proposed, the sourcewise representation

$$\sigma = \int_0^s B(s, s', \{\beta\})\, d\mathbf{e}(s') \tag{2}$$

where $\{\beta\}$ is a set of parameters reflecting the influence of the complex geometry of the deformation process. It was proposed to take the set of curvatures and torsions of the strain trajectory $\{\beta\} = \{\varkappa(s), \varkappa(s')\}$ as such parameters and if necessary their derivatives with respect to the arclength s as well. Certainly, in such a selection of the parameters $\{\beta\}$ the kernel B must be predetermined at points of singularity $\{\varkappa\}$, i.e., at corner points of the deformation trajectory /1/.

As is seen, there is no need to use the yield surface concept in either (1) or (2).

Another example of the theory where the yield surface can also not be utilized is the theory of plasticity based on the hypothesis of local definiteness /4, 5/. Indeed, if the equations of the local definiteness hypothesis are supplemented by an equation describing the evolution of the stress intensity $\sigma_u = |\sigma|$ (omitting the condition $\sigma_u = \Phi(s)$ /5/)

$$d\sigma_u/ds = f(s, \sigma_u, \{\theta_i\}) \tag{3}$$

where $\{\theta_i\}$ are the angles of orientation of the vector σ at the accompanying Frenet point of the deformation trajectory, then we obtain a plasticity theory that takes account of the complexity of the deformation process but does not utilize the yield surface or the unloading condition.

A plasticity theory has been proposed in which the functional was selected in the form /6/

---

*Prikl. Matem. Mekhan., 53, 2, 319-332, 1989

$$\sigma = \int_0^z J(z - z')\, de(z') \tag{4}$$

which is formally analogous to the linear viscoelasticity functional except that it was proposed to use the singular "internal" time z, generally a function of the deformation process, as the parameter of the process rather than the linear physical time t (hence, the theory is called endochronic i.e., with internal time).

The measure of the internal time z was initially defined by the relationship

$$dz = f^{-1}(s)\, ds, \quad ds = |\, de \,| \tag{5}$$

The function $f(s)$ corresponds to effects of the isotropic hardening type, and was consequently called the hardening function. By analogy with the standard rheological models of linear viscoelasticity, the kernel J was selected in a form corresponding to the generalized Maxwell model

$$J(z) = \sum_{i=1}^{N} 2G_i e^{-\alpha_i z} \tag{6}$$

Despite its apparent simplicity, the endochronic plasticity theory (EPT) enabled a number of important features of the elastic-plastic deformation of materials to be described at a qualitative and sometimes even at a quantitative level from a single viewpoint, for example, linear and non-linear hardening (and softening), the retardation phenomenon, non-linear unloading, hysteresis, hysteresis loop stabilization under cyclic loading, the "plunge" of σ_u at the break in the deformation trajectory, etc. /6-8/. Meanwhile, it was clarified that the EPT also possesses certain extraordinary properties. For example, if violates the Il'yushin-Drucker postulate /9/. This results in effects of the cyclic-creep and cyclic-relaxation type. Moreover, a serious quantitative discrepancy between experimental and theoretical data was often observed when comparing the predictions of the theory (4)-(6) with the results of experiments on complex loading. These facts became the basis for a serious criticism of the initial EPT modification /9-14/.

It is easy to comprehend the reason for the discrepancy between EPT predictions and test data by examining the form of the functional (4)-(6) more carefully.

It is quite obvious that the EPT functional is an "oversimplified" modification of the source-wise representation of the vector σ (2) in which the dependence of the kernel on the geometry of the deformation trajectory is omitted, and the arclength s is used as the parameter of the process as before. This results in the deformation increment de, corresponding to both active loading $(\sigma de > 0)$ and passing loading $(\sigma de < 0)$, making an essentially identical contribution to (4) and (5) that certainly cannot be valid in plasticity theory in the general case.

Note that such a contradiction does not occur in the source-wise representation since the geometry of the deformation process is taken into account in the selection of the kernel B.

The remark made in /10/ that three are actually insufficient foundations for the selection of the kernel J in the form (6) and, even more, that the form of the hardening function f and the values of the parameters G_i, α_i is not determined successfully in an independent manner (and with sufficient accuracy) from experiments, can also be appended to the above.

To what extent the EPT function is "oversimplified" as compared with the exact functional (1) can be understood by examining the expansion (approximation) of the functional (1) in the Frechet-Volterra functional-power series /15/

$$\sigma = \sum_{k=1}^{\infty} \int \ldots \int_0^s J_{2k-1}\, de_1 (de_2 \cdot de_3) \ldots (de_{2k-2} \cdot de_{2k-1}) \tag{7}$$

$$J_{2k-1} = J_{2k-1}(s, \xi_1, \xi_2, \ldots, \xi_{2k-1}), \quad de_i = de(\xi_i)$$

It can be shown that the series is reduced to the form

$$\sigma = \int_0^s K(s, \xi)\, de(\xi) \tag{8}$$

up to processes with mean curvature inclusive for expansion in the small parameter.

$\eta = |\varkappa| \lambda$ ($\varkappa$ is the curvature of the deformation process and λ is the retardation trace).

Taking account of the next order of complexity (in the parameter η) of the deformation trajectory results in a functional of the form

$$\sigma = \int_0^s K_1(s, \xi)\, d\mathbf{e}(\xi) + \iiint_0^s K_3(s, \xi_1, \xi_2, \xi_3)\, d\mathbf{e}_1 (d\mathbf{e}_2 \cdot d\mathbf{e}_3) \tag{9}$$

etc. Therefore, strictly speaking, the functional of the initial EPT modification is exact only for deformation trajectories no more complex than a trajectory with mean curvature and it must be considered as approximate for more complex trajectories. This explains the quantitative discrepancy observed between the predictions of this EPT modification and test data.

The disadvantages of the functional of the initial EPT modification can perhaps be eliminated in a different manner. Writing the EPT functional in the form (8), it is possible to avoid the representation of the kernel J by the sum (6) and to define J instead from experiments in complex loading on trajectories in the form of a two-branched broken line. It is also possible to replace the simplest functional (4), (8) by the more complex functional (9). The "accuracy" domain of this functional is substantially wider but two kernels K_1 and K_3, where the latter depends on four arguments in the general case, must already be constructed from experimental data for its definition. Construction of the kernels K_1 and K_3 is simplified somewhat if the body material does not age with respect to s, in which case it can be considered that

$$K_1(s, \xi) = K_1(s - \xi), \quad K_3(s, \xi_1, \xi_2, \xi_3) = K_3(s - \xi_1, \ s - \xi_2, \ s - \xi_3)$$

Although the kernels K_1 and K_3 can, in principle, be constructed from data of complex loading experiments with deformation trajectories in the form of multiple-section broken lines, this is a quite complex experimental problem.

The question of the selection of the plasticity functional is solved differently in /16, 17/ where a parametric representation is proposed of the functional of the relationship between σ and $\mathbf{e}$

$$\sigma = \int_0^\rho L_1(\rho, \rho')\, d\mathbf{R}(\rho') \tag{10}$$

$$\mathbf{e} = \int_0^\rho L_2(\rho, \rho')\, d\mathbf{R}(\rho'), \quad d\rho = f^{-1}(R)\, dR, \quad dR = |d\mathbf{R}|$$

A characteristic feature of such a form of the plasticity functional is that the relation between the vectors σ and $\mathbf{e}$ is not sought directly but in terms of an auxiliary vector $\mathbf{R}$ whose form is not stipulated in advance but it is considered that it reflects the influence of the microstrains and microstresses on the plasticity process. Although the analytic form of the tensor-parametric theory (10) is considerably more complex than the EPT, the presence of two kernels L_1 and L_2 in place of just the one J considerably expands the possibilities of the theory and enables a wider class of material properties to be described.

Valanis chose a completely different means of revising the initial EPT modification. Instead of changing the form of the functional he proposed changing the definition of the internal time measure by introducing a "new" measure /18-20/

$$dz = f^{-1}(\xi)\, d\xi, \quad d\xi = |d\mathbf{e} - \chi E^{-1} d\sigma| \tag{11}$$

Here E is the shear modulus and $\chi \in [0, 1]$ is an additional parameter of the model. The same measure was implicitly used earlier in /21/*. (*Complex loading processes in EPT with a new internal time measure were examined in the paper by Mosolov A.B., "On plasticity theory relationships taking account of deformation process complexity", Moscow State University, 1980. Deposited with VINITI, 2995, July 19, 1980, where a more general form of the new measure was proposed to take account of the dependence of χ on the history of the strain process.).

In order to understand what replacement of the measure (5) by (11) leads to, we consider the simplest EPT model when $J(z) = E\exp(-\alpha z)$. In this case (4) and (11) can be rewritten in the form of trinomial plasticity theory relationships

$$d\sigma = E d\mathbf{e} - \alpha\sigma dz \tag{12}$$

Curves of the dependence of σ on ε are presented in Fig.1 for loading and subsequent unloading in the one-dimensional case corresponding to model (12). The parameters E and α remained constant $f(\xi) = 1 + \beta\xi$, the parameter χ took the values 0, 0.5, 0.95 for the curves *1, 2, 3*, respectively. As follows from the graphs presented, the model under consideration corresponds to a material with linear hardening of the isotropic type.

The quantity dz can be represented in the form

$$dz = \varphi(\xi, \sigma_u, \theta)\, d\xi \tag{13}$$

where θ is the angle between the vectors σ and de. Furthermore, for simplicity, only two-dimensional loading and deformation processes will be examined.

Equations describing the evolution of σ_u and θ

$$d\sigma_u/ds = E\cos\theta - \alpha\sigma_u\varphi = \psi_1(\xi, \sigma_u, \theta)$$
$$d\theta/ds = -\varkappa - E\sigma_u^{-1}\sin\theta = -\varkappa - \psi_2(\xi, \sigma_u, \theta)$$

can be obtained from (12).

These equations are actually identical with the equations of the theory based on the Lenskii local definiteness hypothesis with the difference, however, that σ_u is not considered to be a universal function of s. In particular, a plunge of σ_u occurs at a corner in the deformation trajectory. Drop profiles are shown in the inset to Fig.1 for deformation along a two-section broken line with a corner angle 90° marked by the open circle, the material parameters are as before and the trajectory corner occurs at the point A marked with the open circle $\Delta s = s - s_A$, where s_A is the value of s at the time of the break. It is seen quite well that the depth of the drop depends on the magnitude of the parameter χ where the drop vanishes as $\chi \to 1$.

A comparison between the predictions of model (12) (the dashed line) and the experimental data (the solid line) is shown in Fig.2 for S10C steel /22/. The deformation trajectory had the form of a two-section broken line with a corner angle of 90° $U = \sigma_u\cos\theta$, $V = \sigma_u\sin\theta$, $\chi = 0.65$. Computations according to the initial EPT modification ($\chi = 0$) are shown by the dash-dot curve.

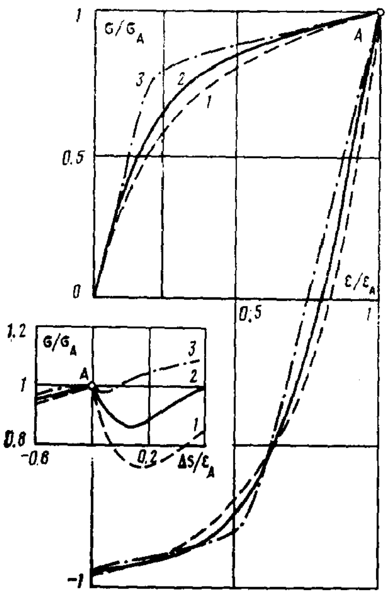

Fig.1

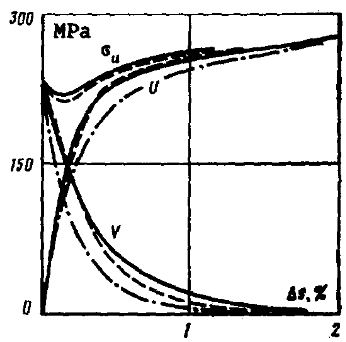

Fig.2

The introduction of the new measure (12) turned out to be useful and permitted a significant extension of the predictive possibilities of the EPT. However, it should be noted that the parameter χ received no physical interpretation and remained essentially a formal parameter (adjustment) of the theory. Moreover, Valanis and his followers /23-25/ took the strict equality $\chi = 1$, i.e., $d\xi = ds_p$, in later publications, where s_p is the plastic arclength (the Odqvist parameter). The yield surface concept occurs in a natural manner for such a definition of the internal time parameter z, and the plasticity functional is written in the form /23/

$$\sigma = \sigma_0\, de_p/dz + \int_0^z J_0(z - z')\, de_p(z')$$
$$de_p = de - E^{-1}d\sigma,\ dz = f^{-1}(\xi)\, d\xi,\ d\xi = |de_p|$$

where σ_0 is the yield point, while J_0 is a non-singular kernel, or in the form /24/

$$\sigma = \int_0^z J(z - z')\, de_p(z'), \quad J(z) = z^{-\alpha} J_0(z), \quad \alpha < 1$$

Therefore, the modern EPT realization that Valanis proposed returns it to the class of plasticity theories with a yield surface although, possibly, not with an unassociated flow law /26, 27/. This can hardly be considered a sequential development of the thoery which was initially of interest precisely in that there was no yield surface in it. The latter assertion is ordinarily stressed especially in research devoted to EPT and, consequently, needs a certain refinement.

Indeed, there is not need to use the unloading condition and the yield surface concept in constructing governing relationships in EPT (there is simply no such surface in a strict sense for $\chi \neq 1$) and the plasticity functional is given by (4) and (11) (or (10)) for all possible deformation trajectories (including even trajectories corresponding to unloading). However, to interpret experimental data and to compare them with other plasticity theory modifications (with flow theories, say), it will often be convenient to introduce the conditional yield surface (CYS) concept into EPT. It is pertinent to recall here that the yield surface is defined in experimental investigations with a certain tolerance on whose magnitude the dimensions, shape and location of the yield surface in stress space can depend considerably. Most often a tolerance in the magnitude of the residual deformations (usually 0.01-0.2%) is chosen.

It is possible to proceed analogously in the EPT, except we shall give the tolerance in the magnitude of the relative plastic deformation increment. This simplifies the calculations significantly in the plasticity theory modification under consideration and corresponds more to the substance of the situation. For clarity, we shall limit ourselves first to the simplest EPT model (12).

We introduce the following definition. Let $0 < \delta \leqslant 1$ be a certain small number (tolerance). We call a quasi-elastic deformation in the tolerance δ a deformation process such that

$$| d\mathbf{e}_p/ds | < \delta, \; d\mathbf{e}_p = d\mathbf{e} - E^{-1} d\boldsymbol{\sigma}, \; ds = | d\mathbf{e} | \tag{14}$$

The domain of stress space at each of whose points condition (14) holds for any deformation process will be called the quasi-elastic deformation domain Ω_δ. We call S_δ the closure boundary of Ω_δ for a CYS in the tolerance δ.

Let us construct S_δ for the model given by (2), from which it follows that

$$d\mathbf{e}_p = \alpha E^{-1} [f(\xi)]^{-1} \mid \chi d\mathbf{e}_p + (1 - \chi)\, d\mathbf{e} \mid \boldsymbol{\sigma} \tag{15}$$

We use the notation

$$y = \left| \frac{d\mathbf{e}_p}{ds} \right|, \quad \sigma_u = |\boldsymbol{\sigma}|, \quad \cos\theta = \frac{\boldsymbol{\sigma}\, d\mathbf{e}}{\sigma_u\, ds}, \quad \varphi = \alpha (Ef(\xi))^{-1}$$

Squaring both sides of (15), we obtain a quadratic equation in y whose root is

$$y(\theta) = (1 - \chi)\varphi\sigma_u \frac{\chi\varphi\sigma_u \cos\theta + \sqrt{1 - \chi^2\varphi^2\sigma_u^2 \sin^2\theta}}{1 - \chi^2\varphi^2\sigma_u^2}$$

(the second root is discarded as extraneous). Eq.(14) acquires the form $y(\theta) < \delta$. By the definition of Ω_δ, this inequality should be valid at any point of Ω_δ irrespective of the value of θ. Consequently Ω_δ is actually defined by the inequality $\max_\theta y(\theta) = y(0) < \delta$ or

$$y(0) = (1 - \chi)\varphi\sigma_u / (1 - \chi\varphi\sigma_u) < \delta$$

We hence find

$$\sigma_u < \sigma_\delta(\xi) = \delta (1 - \chi + \delta\chi)^{-1} E\alpha^{-1} f(\xi) \tag{16}$$

The inequality (16) means that Ω_δ coincides with an open sphere of radius $\sigma_\delta(\xi)$ with centre at the origin in stress space, and in this case the CYS is a sphere of radius $\sigma_\delta(\xi)$.

It follows from (16) that the CYS can change during deformation by expanding isotropically or being reduced in conformity with the behaviour of the function $f(\xi)$. This indeed explains the designation of the function f as the hardening (softening) function. Therefore, model (13) corresponds to material with isotropic hardening ($df/d\xi > 0$) or softening ($df/d\xi < 0$). In the simplest case $f(\xi) \equiv 1$ relationship (14) describes material without hardening with the ultimate value of the stress $\sigma_0 = E\alpha^{-1}$.

It follows from (16) that if $\chi \neq 1$, then $\sigma_\delta < \sigma_0$. The spherical layer $\sigma_\delta(\xi) < \sigma_u < \sigma_0(\xi) = \sigma_0 f(\xi)$ can be called the plasticity layer since condition (14) is generally violated therein, meaning that the plastic deformation increment can become "noticeable", i.e., exceed the tolerance δ.

Unlike classical plasticity theories (of the flow-theory type), the loading point in the EPT can lie in a plasticity layer outside the CYS. This brings the EPT close to the class of two-surface plasticity theories /28, 29/, where the outer boundary of the plasticity layer, the surface $\sigma_u = \sigma_0(\xi)$ in the EPT is analogous to the limit surface (loading surface) in two-surface plasticity theories, and the surface $\sigma_u = \sigma_\delta(\xi)$ (CYS) corresponds to the yield surface.

As follows from (16) and the definition of $\sigma_0(\xi)$, the change in the size of the limit surface is determined by the hardening function, and the size of the CYS depends very much on the magnitude of the parameter χ also. It is assumed in the majority of research on EPT that χ is a constant parameter, but it can be shown that, taking account of the dependence of χ on the history of the deformation (loading) process, in the form $\chi = \chi(\xi)$, say, we can take account of the regularities of the elastic-plastic deformation more correctly and reach better agreement with experimental results.

Curves of the dependence of σ on ε during unloading (for a one-dimensional process), computed by means of (12) for $f = 1$, are represented in Fig.3 in σ/σ_0, $\varepsilon/\varepsilon_0$ ($\varepsilon_0 = \sigma_0/E$) coordinates. The dashed curves correspond to the constant value $\chi = 0.9$, the solid lines are the results of a computation under the simplest assumptions $\chi = 1 - \beta\varepsilon_0^{-1}\xi$, $\beta\varepsilon_0^{-1} \simeq 0.16$; and the points correspond to experimental data for S15C steel /30/.

Taking account of the dependence of χ on the history of the loading process, the CYS deformation and the limit surface can be considered independently.

Within the framework of the EPT it is possible to go from model (13) with isotropic hardening to models with translational hardening (in the CYS sense). This is achieved by replacing the kernel $J \to \mu + J$, where μ is the hardening modulus or by replacing in (2): $\sigma \to \sigma - \mu e$ (analogously $\sigma \to \sigma - \mu e_p$). It can be verified that in this case the inequality (15) takes the form $|\sigma - \mu e| < \sigma_\delta(\xi)$. This means that as before the CYS will be a sphere of radius $\sigma_\delta(\xi)$, but now its centre shifts to the point μe, as it should under translational hardening. The limit surface will similarly be determined by the equality $|\sigma - \mu e| = \sigma_0(\xi)$, and consequently the centre of the whole plasticity layer shifts to the point μe.

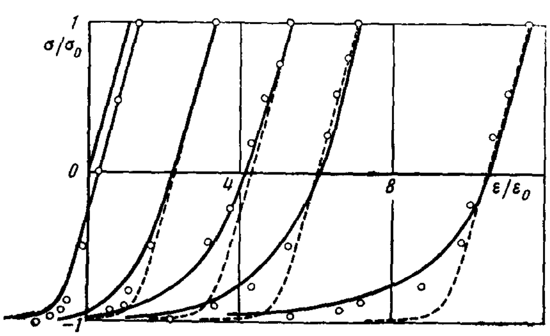

Fig.3

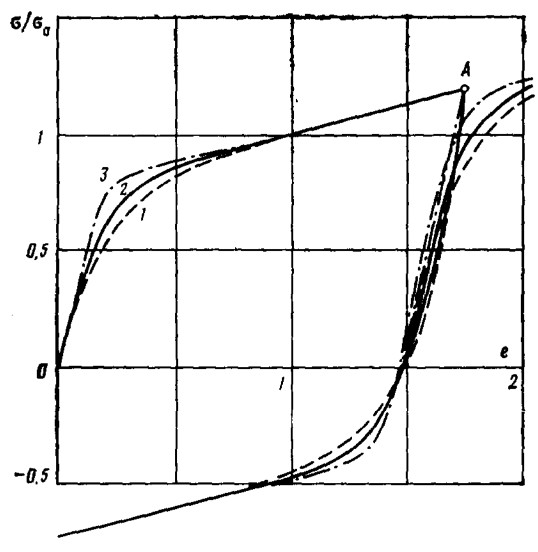

Fig.4

The governing equation of the simplest EPT model with linear tanslational-type hardening (the kernel of the functional (4) is taken in the form $J(z) = \mu + E\exp(-\alpha z)$, $\alpha = E/\sigma_0$) can be writtten as follows

$$d\sigma = (E + \mu)\,de - \alpha(\sigma - \mu e)\,d\xi \tag{17}$$

where μ has the meaning of a hardening modulus while σ_0 is the yield point.

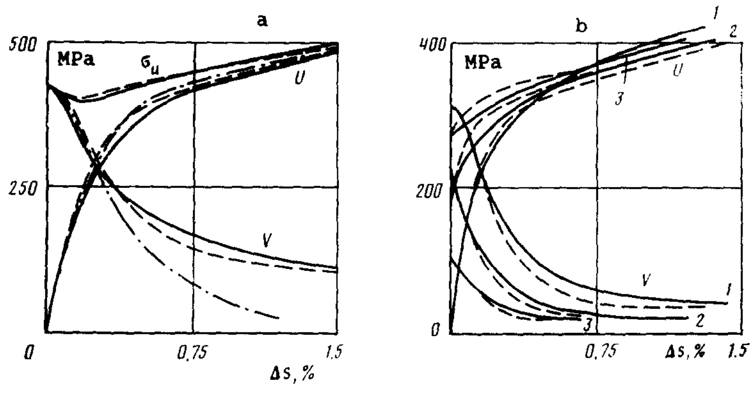

Fig.5

The behaviour of the material described by model (17) under uniaxial loading and subsequent unloading is shown in Fig.4.

The model parameters E, σ_0 and μ are constants and correspond to the parameters of an EPT model with linear isotropic-type hardening whose behaviour is shown in Fig.1. As before, the parameter χ took the values 0, 0.5, 0.95 for curves *1*, *2*, *3*, respectively and $e = \varepsilon E/\sigma_0$.

A comparison of the predictions of model (17) (the dashed lines) with the results of experiments (solid lines) in the complex loading of steel 45 /10/ is shown in Fig.5. The deformation trajectories in the tests were two-section broken lines with corner angle $\theta_0 = 90°$ for $s_0 = 2.16\%$ (for Fig.5a) and $\theta_0 = 23, 46, 90°$ (for Fig.5b), $U = \sigma_u \cos\theta$, $V = \sigma_u \sin\theta$, $\chi = 0.9$. Curves computed on the basis of an EPT model with linear isotropic-type hardening are shown by the dash-dot lines in Fig.5a for comparison.

Allowing a dependence of μ on ξ or considering functionals with more complex kernels, non-linear effects can be taken into account during translational-type hardening.

Numerous experiments show that the yield surface constructed according to a given tolerance can only be displaced in stress space during deformation by isotropically altering its size, but can also change shape noticeably. The transverse dimensions of the yield surface often remain unchanged, i.e., there is no "transverse" Bauschinger effect (the Phillips rule) /31, 32/. However, it should be kept in mind that the Phillips rule is not always satisfied /33/.

The above-mentioned deformation features of the yield surface can be taken into account within the EPT framework by changing the governing relationships somewhat. We again examine model (12) and for simplicity we limit ourselves to loading processes in one plane Σ_2.

Let O be the origin in Σ_2, and $\mathbf{m}_1$ and $\mathbf{m}_2$ orthogonal unit vectors of the basis. We change to polar coordinates with centre at O. Then the location of any point on the loading trajectory will be determined by the stress intensity σ_u and the polar angle θ, measured from the direction $\mathbf{m}_1$, say. We shall assume that the deformation anisotropy occurring in a specimen during deformation can be described by the hardening function f and the parameter χ selected in the form

$$f = f(\xi, \sigma_u, \zeta_\theta(\xi)), \quad \chi = \chi(\xi, \sigma_u, \zeta_\theta(\xi))$$
$$\theta(\xi) = \arccos(\mathbf{m}_1 \boldsymbol{\sigma}(\xi)/\sigma_u(\xi))$$

Here $\zeta_\theta(\xi)$ is a parameter ensuring the dependence of f and χ on the history of the loading (deformation) process. We define it as follows

$$\zeta_\theta(\xi) = \int_0^\xi \beta(\xi, \xi', \cos\gamma^\theta(\xi'))\, d\xi', \quad \cos\gamma_\theta(\xi') = \frac{\mathbf{m}_\theta\, d\boldsymbol{\sigma}(\xi')}{|d\boldsymbol{\sigma}(\xi')|}$$

($\mathbf{m}_\theta$ is a unit vector such that $\mathbf{m}_1 \mathbf{m}_\theta = \cos\theta(\xi)$). Sometimes it is more convenient to use another definition

$$\zeta_\theta(\xi) = \int_0^\xi \beta_1(\xi, \xi', \cos\psi_\theta(\xi'))\, d\xi', \quad \cos\psi_\theta = \frac{\mathbf{m}_\theta\, d\mathbf{e}(\xi')}{|d\mathbf{e}(\xi')|}$$

The specific form of the function f, χ, β, β_1 is determined by the material properties. As before, the CYS motion in stress space can be taken into account by the substitution $\sigma \to \sigma - \mu e$.

Let us consider a simple example. Let the material be described by the equations

$$\begin{aligned} &d\sigma_1 = Ede - \alpha\,[f(\zeta_\theta)]^{-1}\sigma_1 d\xi, \quad \sigma_1 = \sigma - \mu e \\ &d\zeta_\theta = \varphi(\cos\psi_\theta(\xi))d\xi \\ &\varphi(x) = -k_- x h(-x) + k_+ x h(x), \quad 0 < k_+ < k_- \\ &f(x) = 1 - \beta(1 - e^{-\lambda x}), \quad \beta < 1 \end{aligned} \tag{18}$$

where $h(x)$ is the Heaviside function, and we will consider the parameters $\alpha, \beta, \lambda, E, \mu, \chi, k_\pm$ to be constants, where $\beta, \lambda > 0$. Let the deformation trajectory be a two-section broken line with corner angle θ_0 for $s = s_0$ (or for $\xi = \xi_0$). We will measure the angle θ from the direction of the first section. Then for $s < s_0$ we have $\cos\psi_\theta = \cos\theta$, $\zeta_\theta = \varphi(\cos\theta)\xi$, while for $s > s_0$ we have, respectively $\zeta_\theta = \varphi(\cos\theta)\xi_0 + \varphi(\cos(\theta - \theta_0))(\xi - \xi_0)$. To be specific, let $\theta_0 = \pi/2$, $\chi = 0.95$.

The initial yield surface (curve *1*) and the CYS sequence for the loading points *B, C, D, E* are shown by solid lines in the lower right side of Fig.6. The dash-dot line denotes the leading trajectory, the dashed circles are superposed for comparison and corresond to taking account of just the purely translational hardening. The model parameters are selected as follows

$$\beta = 0.8; \quad k_+\lambda = 10^{-3}; \quad k_-/k_+ = 15; \quad E/\sigma_0 = 1, \quad \mu/E = 0.1; \quad \delta = 0.2$$

A curve of the dependence of σ on ε for simple loading is presented in the left upper part of Fig.6 for the material mentioned $\varepsilon_0 = \sigma_0/E$.

The initial yield surface *1* and three successive yield surfaces *2, 3, 4*, constructed after preliminary tension of a thin-walled tubular specimen fabricated from technically pure aluminium mark 1100-0 to stresses of 46.5, 52.6 and 63.6 MPa, respectively /32/, are shown by the solid lines in Fig.7. The CYS corresponding to the model (18) are denoted by dashed lines. It is assumed here that $\zeta_\theta \simeq \pm k_\pm \sigma_0 \mu^{-1} \cos\theta x$ (the plus sign refers to the case $\cos\theta > 0$, and the minus to the case $\cos\theta < 0$), $x = \sigma_0^{-1}\Delta\sigma$, $\Delta\sigma = \sigma_z - \sigma_0$, where $k_+\lambda\mu^{-1}\sigma_0 \simeq 4.5\cdot 10^{-3}$, $k_-\lambda\mu^{-1}\sigma_0 \simeq 2.38\cdot 10^{-2}$, $\chi = 0.95$, $\beta \simeq 0.84$. Only the upper halves of the appropriate curves (for $\tau > 0$) are shown in Fig.7 since the lower halves (for $\tau < 0$) are arranged symmetrically about the σ_z axis.

Graphs of $y_+ = f(\zeta_0)$ and $y_- = f(\zeta_\pi)$ as a function of x are presented in Fig.8 (the upper and lower curve, respectively), the points with the plus and minus signs inside correspond to the experimental data represented in Fig.7.

It is seen from Figs.6 and 7 that the model under consideration satisfies the Phillips rule. Special value should not be attached to the appearance of angular points on the CYS (although certain recent experimental results /34/ possibly favour this) since they are related to the selection of hardening laws in the form (18). The CYS deformation law can be selected so that its smoothness during the deformation process will be conserved. Another approach to the determination of the internal time measure was noted in /11, 35, 36/*. (*See also the previous footnote.)

By selecting the increment EPT mode, Bažant /11/ described the plasticity functional in the form

$$\sigma = F_1 de + F_2\sigma dz, \quad dz = F_3 ds$$

where F_1, F_2, F_3 are functions of σ, e, z. A quite complex specification of these functions, dependent on a large number of parameters /35, 36/, was proposed for mountain rocks and concrete. This would permit better agreement to be obtained between the experimental and computational data for different loading modes, however, determination of the parameters in F_1, F_2 and F_3 from experimental data is quite difficult.

By generalizing the modifications presented above for the selection of z it can be considered that the internal time measure should be selected so that the nature (active and passive) of the deformation processes are correctly taken into account. Consequently, we set $dz = \varphi(s, \{\beta\})\,ds$, where $\{\beta\}$ is the set of parameters ensuring that the "complexity" of the deformation processes is taken into account. The form of the function φ and the parameters $\{\beta\}$ requires additional investigation but keeping the relationship (13) in mind, the generalized local definiteness hypothesis can be used as a simple and sufficiently general assumption and it can be considered that $\{\beta\} = \{\sigma_u(s), \theta(s)\}$, that is

$$dz = \varphi(s, \sigma_u, \theta)\,ds \tag{19}$$

It is natural to assume that φ is an even decreasing function of θ, $0 \leqslant \varphi \leqslant 1$. Taking into account that for almost simple processes the arclength s is a completely adequate parameter at the history, it can be considered that φ also satisfies the condition $\varphi(s, \Phi(s), 0) = 1$, where $\Phi(s)$ is the hardening function under simple loading.

We present an example of utilizing the models (4) and (19) to describe complex loading experiments. Results of test on biaxial deformation of brass are represented in Fig.9 (the open circles correspond to values of σ_u, and the crosses and dark points to values of σ_1 and σ_3) /37/. The deformation trajectory was a three-section S-shaped broken line with round-offs of radius $R = 0.09\%$ at the break points (Fig.10). The non-linear hardening of brass is to be described by the equation

$$\Phi(s) = 106.7 + 51.3\sqrt{s}\,(\text{MPa}),\ s > 0.5\%$$

We use the models (4) and (19) to describe the experiments for

$$J(z) = E\exp(-\alpha z),\quad \varphi(s, \sigma_u, \theta) = \frac{E}{\alpha\Phi(s)}\left(\frac{\sigma_u}{\Phi(s)}\right)^k \cos^{2\nu}\frac{\theta}{2}$$
$$\alpha = E/\Phi(s_0)$$

where $k = 3,\ \nu = 1,\ s \geqslant s_0 = 1.5\%$ and E has the meaning of the elastic modulus. The computed curves are shown in Fig.9 by solid lines.

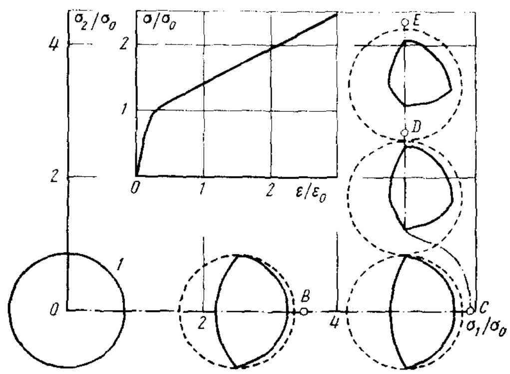

Fig.6

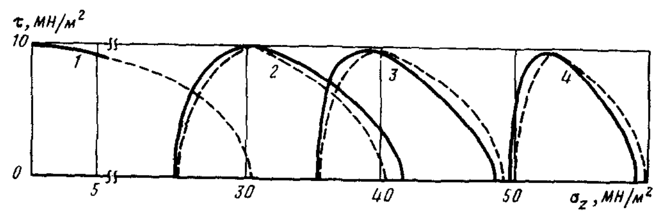

Fig.7

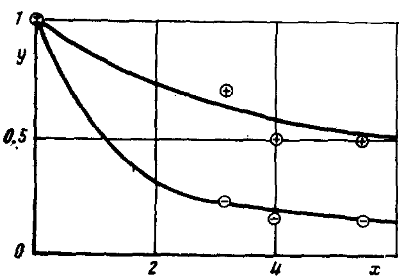

Fig.8

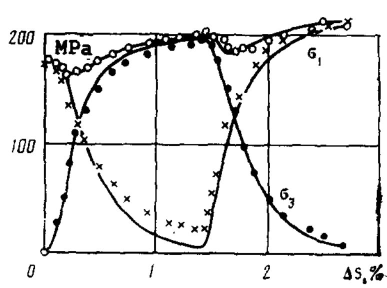

Fig.9

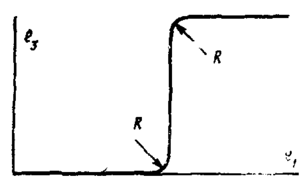

Fig.10

The parameters $\{\beta\}$ can be determined analogously even in source-wise plasticity theory (SPT) (2); then the SPT functional is written in the form

$$\sigma = \int_0^s B(s, s', \sigma_u(s), \sigma_u(s'), \theta(s'))\, de(s') \tag{20}$$

Comparing (4), (19) and (20), the "duality" of these relationships can be noted. They differ only in that the geometry of the deformation processes in EPT is taken into account in determining the internal time parameter, while it is in the selection of the kernel B in the SPT.

It is sometimes assumed that a particular local definiteness property is valid, according to which: if after an arbitrary loading process a break in the deformation trajectory is realized at a certain time $s = s_0$ so that the direction of the vector $\mathbf{p}_0 = d\mathbf{e}/ds\,(s_0 + 0)$ coincides with the direction of the vector $\sigma\,(s_0 - 0)$ and then deformation along a line determined by the vector $\mathbf{p}_0$ is continued, then the vector σ remains parallel to $\mathbf{p}_0$ /38/. In this case it can be shown that the kernel of the EPT and SPT functionals are factorized, i.e., can be represented in the form of the products

$$J\,(z, z') = J_1\,(z)\,J_2\,(z')$$
$$B\,(s, s', \sigma_u\,(s), \sigma_u\,(s'), \theta\,(s')) = B_1\,(s, \sigma_u\,(s))\,B_2\,(s', \sigma_u\,(s'), \theta\,(s'))$$

Because of this the EPT and SPT functionals can be rewritten in the form of a trinomial plasticity relationship /3, 39/

$$d\sigma = Nd\mathbf{e} - M\sigma ds$$

or in the form of equations of the generalized local definiteness hypothesis

$$\frac{d\sigma_u}{ds} = N\cos\theta - M\sigma_u,\quad \frac{d\theta}{ds} = \varkappa - \frac{N}{\sigma_u}\sin\theta$$

where N and M can be functions of z, s, σ_u and θ.

Despite the explicit similarity of the representations of the EPT and SPT functionals, the predictions of their theories can be distinct in a number of cases. The fact is that $N = N\,(z)$ in the EPT while $N = N\,(s, \sigma_u, \theta)$ in the SPT. Consequently, a continuous change in N is predicted for a corner in the deformation trajectory which is in agreement with assumptions expressed earlier about the behaviour of this functional /1, 2, 39/. The situation is different in the SPT since a dependence of N on θ is allowed in this modification of the theory, and therefore N can change by a jump at a corner in the deformation trajectory. There are experimental indications that this corresponds more to reality /40/, although a fairly good approximation for applied computations is ordinarily the assumption according to which N changes weakly during deformation.

REFERENCES

1. IL'YUSHIN A.a., On the relation between stresses and small strains in the mechanics of continuous media, PMM, 18, 6, 1954.
2. IL'YUSHIN A.A., On the principles of the general mathematical theory of plasticity, Questions of Plasticity theory, USSR Academy of Sciences Press, Moscow, 1961.
3. IL'YUSHIN A.A., Plasticity, Principles of the General Mathematical Theory. USSR Academy of Sciences Press, Moscow, 1963.
4. LENSKII V.S., Local definiteness hypothesis in plasticity theory, Izv. Akad. Nauk SSSR, Otd. Tekh. Nauk, Mekhanika i Mashinostroyenie, 5, 1962.
5. LENSKII V.S., On the formulation of a boundary value problem of general plasticity theory Vestnik Mosk. Univ. Matematika, Mekhanika, 6, 1979.
6. VALANIS K.C., A theory of viscoplasticity without a yield surface, Arch. Mech. Stosow. 23, 4, 1971.
7. VALANIS K.C., On the foundations of the endochronic theory of viscoplasticity, Arch. Mech. Stosow, 27, 5/6, 1975.
8. HSIEH B.J., Application of endochronic theory in multidimensional stress states, Nucl. Eng. and Des., 63, 1, 1981.
9. SANDLER I.S., On the uniqueness and stability of endochronic theories of material behaviour. Trans. ASME: J. Appl. Mech., 45, 2, 1978.
10. LENSKII V.S., Modern questions and problems of plasticity in theoretical and applied aspects. Elasticity and Inelasticity, 5, Izd. Mosk. Univ., Moscow, 1978.
11. BAŽANT Z., Endochronic theory of inelasticity and incremental plasticity theory. Mechanics of Deformable Solids: Development Trends, Mir, Moscow, 1983.
12. RIVLIN R.S., Some comments on the endochronic theories of material behaviour, Intern. J. Solids Struct. 17, 2, 1981.
13. VALANIS K.C., On the substance of Rivlin's remarks on the endochronic theory, Intern. J. Solids Struct., 17, 2, 1981.

14. MOSOLOV A.B., Uniqueness in the small and stability in one modification of plasticity theory, Vestnik, Mosk. Univ., Matematika, Mekhanika, 6, 1980.
15. MOSOLOV A.B., On plasticity theory relationships for two-section deformation processes with curved sections, Izv. Akad. Nauk SSSR, Mekhan. Tverd. Tela, 6, 6, 1985.
16. KADASHEVICH YU.I. and MIKHAILOV A.N., On a plasticity theory without a yield surface, Dokl. Akad. Nauk SSSR, 254, 3, 1980.
17. KADSHEVICH YU.I., On new tendencies in the development of flow theory, Problems of the Mechanics of a Deformable Solid, 14, Izd. Leningrad, Gos. Univ., 1982.
18. MOSOLOV A.B. and DINAREV O.YU., On trajectories with angular points in plasticity theory without a yield surface, Probl. Prochnosti, 10, 1986.
19. VALANIS K.C. and READ H.E., A theory of plasticity for hysteretic materials, I. Shear response, Computer and Structures, 8, 3/4, 1978.
20. VALNIS K.C., Fundamental consequences of a new intrinsic time measure: Plasticity as a limit of the endochronic theory, Arch. Mech., 32, 2, 1980.
21. KADASHEVICH YU.I., On different modifications of tensor-linear relationships in plasticity theory. Issledovaniya po Urpugosti i Plastichnosit, 6, Izd. Leningrad, Gos. Univ., 1967.
22. OHASHI Y., Precise measurement and formulation of plastic behaviour of metals. Rozpr. Inz., 26, 3, 1978.
23. WU H.C. and YANG R.J., Application of the improved endochronic theory of plasticity to loading with multiaxial strainpath, Intern. J. Non-linear Mech., 18, 5, 1983.
24. VALANIS K.C., Continuum foundation of endochronic plasticity. Trans. ASME: J. Eng. Mater. Technol., 106, 4, 1984.
25. WU H.C., YANG C.C. and CHU S.C., Further application of endochronic constitutive equation to loading with non-proporitnal axial-torsional strain-path. Intern. J. Non-Linear Mech., 20, 1, 1985.
26. WATANABE O. and ATLURI S.N. A new endochronic approach to computational elasto-plasticity: example of a cyclically loaded cracked plate, Trans. ASME: J. Appl. Mech., 52, 4, 1985.
27. WATANABE O., Anisotropic hardening law of plasticity using an internal time concept. (Deformations of yield surface). Bull. JSME, 30, 264, 1987.
28. KRIEG R.D., A practical two-surace plasticity theory, Trans. ASME, Ser. E., J. Appl. Mech., 42, 3, 1975.
29. PHILLIPS A. and MOON H., An experimental investigation concerning yield surfaces and loading surfaces, Acta Mech., 27, 1-4, 1977.
30. OHASHI Y. and KAWASHIMA K., Analysis of unloading on a thin circular plate of mild steel using actual unloading stress-strain relation, ZAMM, 50, 8, 1970.
31. PHILLIPS A., Experimental plasticity. Some thoughts on its present status and possible future trends. Mech. Plast. Solids. 2, Noordhoff Intern. Publ., Leyden, 1974.
32. PHILLIPS A. and TANG J.-L., The effect of loading path on the yield surface at elevated temperatures, Intern. J. Solids. Structures. 8, 4, 1972.
33. SHIRATORI E., IKEGAMI K. and KANEKO K., Subsequent yield surface determined in consideration of the Bauschinger effect, Mech. Plast. Solids, 1, Noordhoff Intern. Publ., Leyden, 1973.
34. GUPTA N.K. and LAUERT H.A., A study of yield surface upon reversal of loading under biaxial stress, ZAMM, 63, 10, 1983.
35. BAŽANT Z.P. and BHAT P.D., Endochronic theory of inelasticity and failure of concrete, J. Eng. Mech. Div. Proc. ASCE, 102, 4, 1976.
36. BAŽANT Z.P. and BHAT F.D., Prediction of hysteresis of reinforced concrete members. J. Struct. Div. Proc. ASCE, 103, 1, 1977.
37. OHASHI Y., TOKUDA M., MIYAKE T., KURITA Y. and SUZUKI T., Stress-strain relation of integral type for deformation of brass along strain trajectories consisting of three normal straight branches, Arch. Mech., 32, 1, 1980.
38. MALYI V.I., On the simplification of the functionals of the theory of elastic-plastic processes, Prikl. Mekhan., 14, 2, 1978.
39. VASIN R.A. and IL'YUSHIN A.A., On a representation of the elasticity and plasticity laws in plane problems, Izv. Akad. Nauk SSSR, Mekhan. Tverd. Tela, 4, 1983.
40. VASIN R.A., On an experimental confirmation of a law describing the vector properties of plastic materials. Strength, Plasticity, and Viscoelasticity of Materials and Structures, Ural Scientific Centre, USSR Academy of Sciences, Sverdlovsk, 1986.

Translated by M.D.F.